Frege Against the Formalists

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Early formalism can be characterized as an attempt to sidestep the many ontological and metaphysical difficulties that arise in other philosophies of mathematics. Early formalism can be broadly divided into two positions, that of the term formalist, and that of the game formalist. Where the game formalist ”seeks to banish all semantic notions from mathematics”, the term formalist attempts to ”reduce any such notions to purely syntactic ones” (Linnebo, 2017, p. 73). We will first briefly walk through the ideas of the early formalists and present the ideas of their most notable opponent, before concluding with a look at a slightly more modern take on formalism.

The view of the term formalist is that singular terms in mathematical expressions refer to themselves rather than the numbers they denote. The upshot of this is that we then have no reason to doubt the existence of a number whatsoever, since once we write a symbol we have no reason to doubt the existence of the symbol. So, finally we have no reason to doubt what the symbol denotes. Thus we can assert that numbers exist with absolute confidence.

Now, the term formalist seeks to define functional symbols of arithmetic, such as '+' or '×'. We define these based on rewrite rules[1]. These rewrite rules allow us to give mathematical formulas meaning. For example, we can show that '$5 + 5 = 10$' by showing that we can reduce '$5 + 5$' to '10' via our rewrite rules. We can generalize this to say that the truth of an arithmetical expression, say '$a = b$', is really just the ability to transform (via rewrite rules) one term into the other.

Where the term formalist seeks to reduce semantics notions within mathematics, the game formalist seeks to eliminate them entirely. This view distinguishes itself from term formalism by claiming that arithmetic is not a theory of symbols, but rather a game of manipulations played with such symbols. The game formalist sees mathematics as a game, similar to chess, that we play with empty syntactical expressions. This means that numbers are meaningless within the game, beyond the properties assigned to them with respect to the rules of the game. A parallel can be drawn with chess, where pieces on a board have no meaning beyond the rules that we assign to them that dictate their behaviour within the game.

The flavour of formalism peddled by the early school of formalists is not one that was popularized by its proponents, but rather by its opponents. The most notable work on early formalism is a take-down job by the great Gottlob Frege in *The Foundations of Arithmetic* (1903). In it, he attacks two notable defendants of early formalism: game formalist J. Thomae and term formalist E. Heine. Frege outlines three main points in his criticism of Thomae and Heine in *The Foundations of Arithmetic*. These are: (i) Arithmetic has applications that cannot be justified if we consider it to be 'empty'; (ii) Thomae blurs the line between the 'formal arithmetic/game' and the metatheory of the game; and (iii) Neither Heine nor Thomae are able to offer an explanation of infinity within their theories. We will now detail Frege's criticisms of Thomae and Heine before finally turning to Haskell Curry's version of formalism to see how it stands up against Frege.

The applicability of arithmetic is one that Frege hones in on as a particular weak spot of Thomae's game formalist argument.

> Why can no application be made of a configuration of chess pieces? Obviously, because it expresses no thought.... Why can arithmetical equations be applied? Only because they express thoughts.... Now, it is applicability alone which elevates arithmetic from a game to the rank of a science. So applicability necessarily belongs to it. (*Frege Against The Formalists,* §91)

[1] These rewrite rules can be found in Linnebo, 2017, p. 74

Let us first examine what applicability means in this context. M. Resnik makes the claim that Frege is speaking of "inferential applications", let us proceed with this assumption. In this context, "inferential applications" is taken to mean chains of reasoning that allow us to conclude some truth, where that truth has some analogy to reality. Take the example provided by Resnik (p. 62), where we begin with some knowledge: "A 10 foot by 6 foot floor" and "area in square feet equals length in feet times width in feet", which allows us to conclude that (1) '$10 \times 6 = 60$', and so (2) 'area in square feet = 60'. This is an example of applying arithmetic wherein we are expressing a true statement, $10 \times 6 = 60$, that has analogy to reality, the reality being that a 10 by 6 floor has an area of 60 square feet. But in Thomae's conception, our concluding equation is *empty*. How can $10 \times 6 = 60$ be both empty and have an analogy to reality?

It is clear there's an issue here. If Thomae wants his expressions to be empty, he cannot allow them to be empirically verifiable. For, if this were the case, it is clear that there is some meaningful arithmetic being supposed within Thomae's system. But analogy is inescapable, because as Frege says, "applicability necessarily belongs to [arithmetic]" (Frege, §91).

Thomae's game formalism, at various points, appears to suffer from lack of clarity. The place that Frege sees the most issue is within the lack of distinction between the game of mathematics and the *theory* of the game of mathematics (or the "metatheory"). Frege accuses Thomae of presupposing meaningful arithmetic in his formal arithmetic. Thomae lists a handful of rules for his arithmetic, including commutativity, associativity, and so forth. But this does not make sense, says Frege. By listing a rule, such as $a + a' = a' + a$, we encounter many questions. The first for Frege is whether we are to treat each symbol as a sign or only as "figures", as Frege calls them. For, if we treat them as mere figures, stating such a rule is equivalent to showing a chess position to someone with no knowledge of the game in order to explain a rule of chess. If, on the other hand, we treat symbols as referential signs, we end up with a rule of meaningful arithmetic, rather than a formal expression! And so, Thomae's formal arithmetic is not a formal arithmetic at all.

A final simple, but important, retort is that of infinity. Frege makes a "homely objection":

> In order to produce [an infinite series] we would need an infinitely long blackboard, an infinite supply of chalk, and an infinite length of time. (Frege, 1903, §124)

Heine's theory does not have a strong defense against this argument, Frege claims that Heine is victim of a "curious fate: the tangibility of numbers" (ibid. §124). For, if numbers are taken to exist only by their tangibility, how can we claim that any irrational number exists? On the other hand, Thomae provides a slightly more sophisticated approach to the issue of infinities, providing a rule by which to call a sequence infinite. But Frege still finds flaw with this, because Thomae relies upon the notion of a sequence *possibly* being continued indefinitely. But of course it is not possible for humans to inscribe symbols indefinitely. Neither Heine nor Thomae can coherently explain infinities within their conception. This is a major flaw. Frege concludes that defining such notions based on human capacities is a futile endeavour.

In contrast to the game and term formalism of Thomae and Heine, Haskell Curry's formal systems, presented in *Outlines of a Formalist Philosophy of Mathematics* (1951), offer a possible way out of the many problems concerning Thomae and Heine. Although, Curry's formalism is not completely free of issues and raises some important ontological questions that Curry shies away from answering in his thesis. We will first discuss whether Curry's formalism is able to withstand the arguments presented by Frege against Thomae and Heine before turning to the broader ontological questions posed by Resnik in *Frege and the Philosophy of Mathematics* (1980).

The major advantage to Curry's theories as opposed to Thomae and Heine's, and one that Resnik immediately points out is the "precision with which [Curry] formulates his theory" (ibid., p. 69). Resnik provides a laundry list of advantages to Curry's formalism over the early formalists, but chief among them are the lack of reliance on contentful mathematics, the preservation of meaning within mathematics, and a clear definition of truth. One will note that Curry's formalism also allows us to sidestep the question of infinity that Frege presented the early formalists simply by defining the notion of an infinite sequence as internal to some theory. As well, since Curry preserves the "meaning of mathematics", he is able to avoid questions posed by Frege concerning 'applicability'.

So, it seems that Curry's formalism does a very good job avoiding issues pointed out by Frege some 50 years earlier. One may look to developments in metamathematics between Frege's writings and Curry's thesis as a reason for the apparent fortitude of Curry's work. But, Curry's formalism is not without flaws, and in fact, there is an important ontological question that I think discounts Curry's formalism as a viable philosophy of mathematics.

Curry was rather dismissive with regards to the ontological questions concerning his brand of formalism.

> It is unnecessary to inquire further into the meaning of a formal system. It is characteristic of mathematics that it considers only certain essential properties of its objects, regarding others as irrelevant. One of these irrelevant questions is that of the ontology of a formal system. The question of which representation is the real or essential one is a metaphysical matter with which mathematics has no concern. (Curry, 1951, 30 - 31)

But in fact, a major flaw of Curry's system quickly becomes evident when one begins asking such questions. At the core of Curry's theory is the idea that mathematicians are doing no more than showing certain propositions are true in certain formal systems. But this quickly raises the question, by what logic "such and such is a theorem of such and such formal system" (Resnik, p. 65)? Or, more precisely, in what logical system are we able to claim that "formal system X proves proposition Y"? Curry doesn't commit to a clear ontology here, and in fact goes so far as to call any ontological commitment "irrelevant". This allows us to claim that the logic by which we claim "B proves C" is *also* a formal system! Now, we have the claim that "system A proves system B proves C". It is no big leap to see that we will quickly end up with infinitely many "metatheories" proving claims about other metatheories with this logic. This seems like an issue, for if all we have is infinitely many metatheories, any proof is really just a formal deduction within some metatheory. Curry has not provided any reasoning for us to believe that we can derive any real truth from our formal systems. The fact that Curry doesn't present a resolution, and goes so far as to dismiss such questions as irrelevant, should be quite distressing.[2]

It is clear that while some brands of formalism are able to avoid some metaphysical, ontological, and other difficulties that are common in other sorts of philosophies of mathematics, they are definitely not without flaw. Both game and term formalism received quite a severe take down from Frege. And while Curry's work seemed to avoid Frege's criticisms, it is not without it's own issues. To me, formalism still seems like a interesting and useful philosophy, and while more attractive philosophies such as game formalism are fraught with problems, one may wonder whether there is a variety of formalism that could serve as a viable philosophy of mathematics.

[2]I will briefly add that Curry provided something of a response to the final criticisms presented here in the brief note *Remarks On The Definition and Nature of Mathematics*. As far as I am concerned, this article does nothing to refute any of the points made here, or by any of Curry's critics. Curry does not discuss truth in any meaningful way, and I believe "acceptability" is a cop-out to dealing with issues of truth. As such, his article does not resolve any issues originally discussed.